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LETTER TO THE EDITOR

## Percolation in two-dimensional quasicrystals by Monte Carlo simulations

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**Abstract.** We examine whether or not two-dimensional quasi-periodic lattices belong to the same 'universality' class as periodic lattices, and we study bond and site percolation in Penrose tiling and its dual lattice by making use of Monte Carlo simulations. For the sake of comparison, we also investigate percolation in periodic systems such as a square, Kagomé and dice lattice. For all these lattices, we evaluate several critical exponents such as  $\alpha$  related to the total number of clusters,  $\beta$  related to the percolation strength,  $\gamma$  related to the mean cluster size,  $\nu$  related to the correlation length,  $\tau$  related to cluster sizes, and the fractal dimension  $D$ . Our results indicate that universality holds in two-dimensional lattices with or without periodicity where the coordination number could be either single-valued or multi-valued.

Most previous studies of percolation have been concerned with periodic lattices with single-valued coordination. In particular, it is only for these lattices that the so-called 'universality' has been numerically shown to hold where 'universality' indicates that critical exponents are dimensional invariants being independent of the details of lattices (Stanley 1971, Shante and Kirkpatrick 1971, Essam 1980, Stauffer 1985, Lu and Birman 1987). Our purpose therefore is to see whether 'universality' exists also in lattices without periodicity and/or without single-valued coordination. To this end, we calculate several critical exponents of percolation in two-dimensional (2D) quasi-periodic lattices such as Penrose tiling and its dual lattice as well as periodic lattices such as square, Kagomé and dice lattices. Let us note that the coordination number  $z$  in a square, Kagomé and dual lattice of Penrose is single-valued with  $z = 4$  while  $z$  is multi-valued in Penrose tiling ( $z = 3, 4, 5, 6$  and  $7$ ) and in a dice lattice ( $z = 3$  and  $6$ ) with the average  $\bar{z} = 4$ .

We first carry out Monte Carlo (MC) simulations of percolation as follows (Yonezawa *et al* 1989, Sakamoto *et al* 1989). We start with a lattice composed of  $N$  bonds or sites depending on whether the object is bond or site percolation. Out of these  $N$  elements—bonds or sites—we choose  $M$  elements at random, and assume that these  $M$  elements are intact while the other elements are broken. The concentration  $p$  of intact elements is given by  $p = M/(M + N)$ .

At this point, it is worth mentioning the following point. As for Penrose tiling and its dual lattice, we pick up a portion of a given  $N$  such that the frequency of each vertex is almost equal to that for the corresponding infinite system. When this consideration is taken, the dependence of the results on a choice of origin becomes practically insignificant.

A cluster composed of  $s$  connected intact elements is named an  $s$ -cluster. If there exists a cluster which extends from one side of the lattice to the other, say from top to bottom or from left to right, this cluster is said to be 'percolating'.

For each concentration  $p$  of intact elements, we perform  $n$  runs of MC simulations and count the number  $m$  of runs in which there exists a percolating cluster. The ratio  $m/n$  is interpreted to be the probability  $R_N(p)$  of finding a percolating cluster at  $p$  in a lattice of size  $N$ . There are some arguments to assert that  $dR_N(p)/dp$  is approximated by a Gaussian function characterised by the mean  $p_c(N)$  and the standard deviation  $\Delta_N(p)$  (Stauffer 1985, Efros 1986). We have proposed in the preceding letter (Sakamoto *et al* 1989) a new method to estimate the percolation threshold  $p_c$  from the extrapolation of  $p_c(N)$  to infinity. The numerical calculation of this letter is based upon the reliable values of  $p_c$  obtained through our method. On the HITAC M682H (a scalar processor), one MC run of size  $N \approx 100\,000$  bonds takes 0.5 second and the CPU time is nearly proportional to  $N$  in our method.

In the evaluation of critical exponents for percolation, we make use of the cluster-number analysis due to Stauffer. In what follows, we give the definitions of critical exponents and our numerical results (Yonezawa *et al* 1988, 1989).

(i) Exponent  $\nu$  is related to the critical behaviour of the correlation length  $\xi$  in the form  $\xi \propto |p - p_c|^{-\nu}$ . In a finite system of size  $N = L^2$ , the correlation length  $\xi$  becomes of the order  $L$ . Then, it follows that

$$|p_c(N) - p_c| \propto L^{-1/\nu}. \quad (1)$$

The standard deviation  $\Delta_N$  of  $dR_N(p)/dp$  also depends on  $L$  as

$$\Delta_N \propto L^{-1/\nu} \quad (2)$$

so that the exponent is calculated from the slope of the plot of  $\log \Delta_N$  against  $\log L$ . In figure 1, the log-log plots of (2) are presented for bond percolation in the above-mentioned five lattices. The values of  $\nu$  thus estimated are listed in the first column of table 1 in relation to the theoretically predicted value  $\nu_0 = \frac{4}{3}$ . For the bond problem, the deviation of  $\nu$  from  $\nu_0$  is 2% at most, while for the site problem the largest deviation is 6%. When we remember the ways in which  $\nu$  is estimated, we can say that these deviations are small enough and we conclude that the exponent  $\nu$  for either bond or site percolation of any lattice takes the theoretically proposed value (den Nijs 1979, Nienhuis *et al* 1980, Pearson 1980, Blöte *et al* 1981).

(ii) The fractal dimension  $D$  of a percolating cluster at  $p = p_c$  is defined by

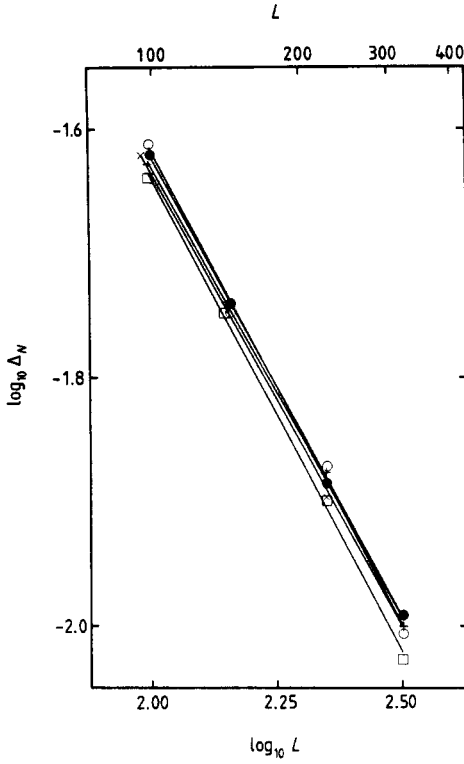
$$s_{\text{perc}} \propto L^D \quad (3)$$

where  $s_{\text{perc}}$  is the size of the percolating cluster at  $p = p_c$ . Therefore,  $D$  is calculated from the slope of the plot of  $\log s_{\text{perc}}$  against  $\log L$  as shown in figure 2. The values of  $D$  thus obtained are presented in the second column of table 1 in relation to the theoretically predicted value  $D_0 = \frac{91}{48}$ . The deviation from  $D_0$  is at most 2% for bond percolation and 3% for site percolation, which undoubtedly shows that the fractal dimension  $D$  is a dimensional invariant.

(ii) Exponent  $\tau$  is related to the number  $n_s(p_c)$  of  $s$ -clusters at  $p = p_c$  by

$$n_s(p_c) \propto s^{-\tau} \quad (4)$$

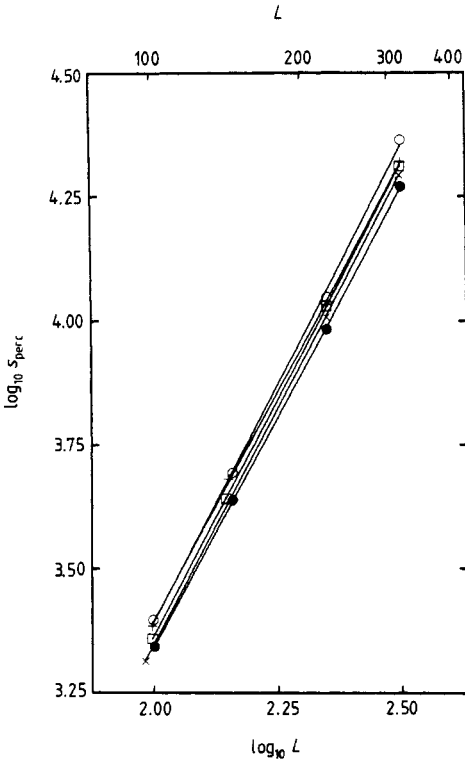
from which it is obvious that the slope of the plot of  $\log n_s(p_c)$  against  $\log s$  gives  $(-\tau)$ . This plot for bond percolation in Penrose tiling is illustrated in figure 3. The straight line in the figure is drawn to fit those data points that give the maximum slope. For small  $s$ , the data points are lower than the straight line since the simple power



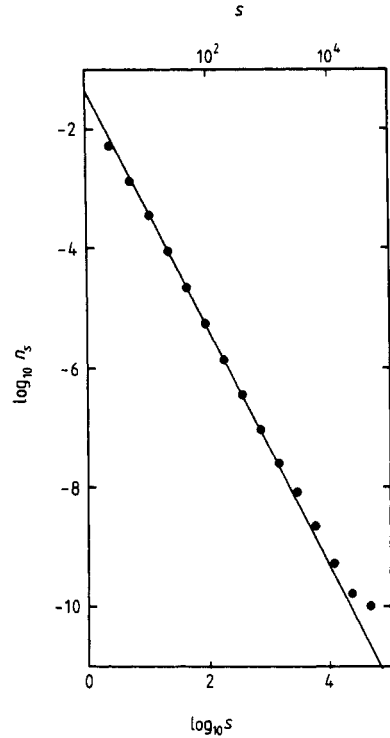
**Figure 1.** Log  $\Delta_N$  plotted against log  $L$  for bond percolation where  $\Delta_N$  is the standard of  $dR_N(p)/dp$ . Plots are given for the square ( $\square$ ), Penrose ( $\bullet$ ), dual of Penrose ( $\circ$ ), Kagomé ( $+$ ) and dice ( $\times$ ) lattices.

**Table 1.** Values of some critical exponents obtained from our simulations.

Percolation type	Lattice	$\nu/\nu_0$	$D/D_0$	$\tau/\tau_0$	$\frac{2-\alpha}{2-\alpha_0}$	$\beta/\beta_0$	$\gamma/\gamma_0$
Bond	square	1.01	1.00	0.96	0.95	0.95	0.92
	Kagomé	1.02	0.98	0.96	0.95	0.76	1.03
	dice	1.01	1.01	0.97	0.94	0.94	0.95
	Penrose	1.01	0.98	0.97	0.94	0.79	1.03
	dual of Penrose	0.98	1.02	0.96	0.95	0.90	1.04
Site	square	1.06	0.96	0.96	1.00	1.02	0.93
	Kagomé	1.01	0.99	0.96	0.99	0.78	0.95
	dice	1.01	1.02	0.95	1.00	1.09	0.88
	Penrose	1.01	1.00	0.94	1.00	1.09	1.05
	dual of Penrose	1.02	1.02	0.97	0.99	0.73	1.16
		$\nu_0$	$D_0$	$\tau_0$	$2-\alpha_0$	$\beta_0$	$\gamma_0$
<b>Theoretical prediction</b>		1.333 ( $=\frac{4}{3}$ )	1.896 ( $=\frac{91}{48}$ )	2.055 ( $=\frac{187}{91}$ )	2.667 ( $=2 - (-\frac{2}{3})$ )	0.139 ( $=\frac{5}{36}$ )	2.389 ( $=\frac{43}{18}$ )



**Figure 2.**  $\log s_{\text{perc}}$  plotted against  $\log L$  at  $p = p_c$  for the bond problem. The value of  $s_{\text{perc}}$  is determined by taking the average of the results obtained from 500 MC simulations at  $p = p_c$ . Plots are given for the square ( $\square$ ), Penrose ( $\bullet$ ), dual of Penrose ( $\circ$ ), Kagomé ( $+$ ) and dice ( $\times$ ) lattices.



**Figure 3.**  $\log n_s$  plotted against  $\log s$  for the bond problem in Penrose tiling. The value  $n_s$  is determined by taking the average of the data from 100 MC simulations in a lattice composed of approximately 400 000 bonds at  $p = p_c$ .

laws of the scaling theory are valid only for large  $s$ . For very large  $s$ , the data points become higher than the straight line since the clusters of larger sizes are cut into several pieces by the existence of the boundaries.

The values of  $\tau$  thus calculated are shown in the third column in table 1 in relation to the theoretically predicted value  $\tau_0$ . The deviation from  $\tau_0$  is at most 2% for bond percolation and 4% for site percolation. Here again, agreement is remarkable and  $\tau$  is a dimensional invariant.

(iv) Exponents  $\alpha$ ,  $\beta$  and  $\gamma$  are respectively related to the total number  $M_0$  of clusters, the percolation strength  $P$  and the mean size  $S$  of finite clusters as follows:

$$M_0(p) \propto |p - p_c|^{(2-\alpha)} \tag{5}$$

$$P \propto (p - p_c)^\beta \tag{6}$$

$$S \propto |p - p_c|^{-\gamma}. \tag{7}$$

When  $p = p_c(N)$ , we can use (1) to eliminate the  $(p - p_c)$  terms in (5)-(7) so that

$$M_0(p_c(N)) \propto L^{-(2-\alpha)/\nu} \tag{8}$$

$$P \propto L^{-\beta/\nu} \tag{9}$$

$$S \propto L^{\gamma/\nu}. \tag{10}$$

Here again, the log-log plots of these three equations give exponents  $(2-\alpha)$ ,  $\beta$  and  $\gamma$  when  $\nu$  is known.

On assuming that  $\nu$  is  $\frac{4}{3}$ , the values of these exponents are calculated as shown in the last three columns in table 1. As for  $(2-\alpha)$ , the deviation from the theoretically proposed value is at most 6% for bond percolation and 1% for site percolation. The percolation strength  $P$  is the probability that a bond or site belongs to a percolating cluster. Then we have  $P = s_{\text{perc}}/L^2 = L^{(D-2)}$  so that  $\beta = \nu(2-D)$ . Since the fractal dimension  $D$  is slightly below the physical dimension  $d=2$ , the accuracy of  $\beta$  is expected to be lower than that of  $D$ . Exponent  $\gamma$  suffers from a similar situation. On noting this point, agreement with the corresponding theoretical values is reasonable both for  $\beta$  and  $\gamma$ , thus indicating that they are also dimensional invariants.

From the analysis presented in the above, we conclude that the scaling hypothesis is valid and 'universality' is fulfilled irrespective of the class of the problem (bond or site percolation), irrespective of the type of lattice (periodic or non-periodic), and irrespective of the kind of coordination therein (single-valued or multi-valued).

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